NUMBER LANGUAGE

A DIFFERENT WAY OF TEACHING ARITHMETIC AT ELEMENTARY SCHOOL

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Numerical language, as any colloquial language, has its own syntax and semantics, which can be understood at early stages of elementary school. In this talk we present numerical language, using properties inherited from the field of polynomials. The central aim for mathematics education is to introduce in elementary teaching, classical concepts from a modern point of view. We try to do it through a not conventional approach which includes an elementary method to check primality and finding prime factors of a given integer without the use of division.

Resumen

El lenguaje numérico, como todo lenguaje coloquial, tiene sintaxis y semántica propias, que pueden entenderse desde los primeros años de la escuela elemental. En esta charla, presentaremos el lenguaje numérico mediante propiedades heredadas del cuerpo de los polinomios. El objetivo central de la educación matemática apunta a integrar en la enseñanza elemental, conceptos clásicos aunque con un enfoque matemático moderno, y aquí buscamos hacerlo a través de una metodología no tradicional, que incluye un método elemental de verificar primalidad y de hallar, sin el recurso de la división, los factores primos de un número

SYNTAX AND SEMANTICS OF NUMBER LANGUAGE

Introduction

A couple of years ago, Yitang Zhang renewed inside the math community the interest for twin primes through his paper related to bounded gaps for primes. Twin primes are pairs of prime numbers of the form p, $p + 2$, spread irregularly among positive integers. From time to time they appear in *prime decades*, such in: 11, 13, 17, and 19; 101, 103, 107, 109; 3461, 3463, 3467, 3469; etc.

Prime decades, Fermat's Last Theorem, the Beal conjecture can be understood by kids, at most at 5^{th} grade, whenever we teach them arithmetic, beginning with number language, its syntax and semantics, to be explained along this talk.

Number language is an attempt to capture the advantages of learning a colloquial language to integrate them in teaching arithmetic. Beginning with an alphabet and some simple syntactic rules, we arrive to a language, where, we can make our math discourse, the same way as we make some colloquial statements.

The main idea behind this approach is the using of polynomials of one variable to introduce a new definition of natural numbers, via the Frege approach of equivalence classes. With this definition we can classify numbers as three disjoint classes, 0 and 1, composite numbers and prime numbers.

The Number Language.

We start with an alphabet, the *digits*, whose symbols are: {0,1,2,3,4,5,6,7,8,9}, whenever we need a decimal representation, and with some primitive parameters like "+" , " · ", and all the machinery inherent to formal languages, as suggested in (Pareja-Heredia 2014).

The words in this language are strings of symbols, where the alphabet ciphers are linked by the primitive parameters with the syntactic rule,

$$\boxed{x^n = 10x^{n-1}} \qquad (*)$$

The meaning of (*) is: the value of n-th position, counted from 0 to n and from right to left is ten times the value of (n-1)-th position. This is the reason why our number system is a decimal positional system. If we would use the alphabet {0, 1} with the syntactic rule, $x^n = 2x^{n-1}$, our numerical system would be a binary one,

So, a natural number a, written with its digits, as $(a_n a_{n-1} \ldots a_1 a_0)$, looks like:

$$a = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 x^0 = \sum_{j=0}^{j=n} a_{n-j} x^{n-j} .$$

When $x = 10$ we recover the conventional form of representing $a = (a_n a_{n-1} \ldots a_1 a_0)$. This is the standard polynomial representation of a. Note that the coefficients in the polynomial are the digits in the decimal representation of a.

The Main Definition

The same way as we define real numbers as equivalence classes of Cauchy sequences, or Frege defined number as equivalent classes of sets, we may define natural numbers as equivalence classes, this time, of equivalent polynomials, via the syntactic rule (*). More precisely, if p and q are polynomials, we say that, $p(x) \equiv q(x)$(module a), if and only if $p(10) = q(10)$.

For instance, $a = 236$ can be associated to the standard polynomial $p(x) = 2x^2 + 3x + 6$, where the coefficients of p are the digits 2, 3, 6. Using syntactic rule (*) on p, we get an equivalent polynomial $q(x) = x^2 + 13x + 6$, so,

$2x^2 + 3x + 6 \equiv x^2 + 13x + 6$ (module a), if and only if, $p(10) = q(10)$.

Namely, the true value of the right side, validate the left side, as we can easily check.

For each number a, we can construct a family of equivalent polynomials, via the syntactic rule (*).

Now we can establish the main definition.

Main Definition. We define number $a = (a_n a_{n-1} \ldots a_1 a_0)$ as the family Ω, of equivalent polynomials module a:

Ω=**{** $\sum_{j=0}^{j=n} b_{n-j} x^{n-j}$ **,** such that, every pair of polynomials p, q in the class, satisfies p (10) =q (10)**} (**)**

In this case the coefficients b_k are not necessarily digits. This class begins with $\sum_{j=0}^{j=n} a_{n-j} x^{n-j}$ **,** the standard representation of a, and ends with a itself (when all coefficients are zero, but $a_0 = a$).

Along this presentation we'll define concepts such as: addition, multiplication and factoring with the resource of the main definition.

A very important definition is related with prime and composite numbers. The classical definition of prime number is linked to the concept of division. Here we take another road: we use factoring.

Definition. A number a, defined as in (**), is said to be *composite* if there exists a polynomial in Ω, which can be factored. Otherwise a will be called *prime*.

THE MAIN RESULT

A consequence of the main definition above is a statement that we call *Primality and factoring Test*. This is a heuristic method of checking if a given number is, or not, prime. Traditionally we have learned to check primality through repeated divisions. In our case we try to reverse the product: $rs = a$; namely, if we know a, the algorithm tries to recover r, s, whenever, that is possible.

The factoring problem is known historically as a very hard one. We can see nowadays, famous number theorists (Friedlander, J. 2015), (Granville, A. 2004)) writing about these matters. However, and it seems paradoxical, we are suggesting to teach at elementary school factoring and prime checking. The reason is that, if kids are able to understand numbers as a language they probably can understand what is under the number concept: addition and multiplication, both operations learned early and easily in the childhood.

From here on, we use, "$p(x) = q(x)$", instead the right symbolism, "$p(x) \equiv q(x) \pmod{a}$". A number $a = (a_3a_2a_1a_0)$ with standard representation $a_3x^3 + a_2x^2 + a_1x + a_0$, can be set as, $(a_3a_2)x^2 + a_1x + a_0$, using the syntactic rule. For instance, $3245 = 3x^3 + 2x^2 + 4x + 5 = 30x^2 + 2x^2 + 4x + 5 = 32x^2 + 4x + 5$.

Primality and Factoring Test

A number $a = (a_3a_2a_1a_0)$, is prime if and only if none of the quadratic polynomials, $(a_3a_2)x^2 + a_1x + a_0$, $((a_3a_2)-1)x^2 + b_1x + b_0$, … , $[(a_3a_2)/2]\, x^2 + t_1x + u_0$, can be factored.

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$[(a_3a_2)/2]$ is the greatest integer less or equal to $(a_3a_2)/2$. The coefficients b_1, b_0, …, t_1, and, u_0 are the result of using iteratively the syntactic rule (*). So, to check primality of four digits numbers, we need only to check whether the quadratic polynomials with coefficients descending from (a_3a_2) to $[(a_3a_2)/2]$ can be factored.

If we get all the factors for a polynomial in Ω, they are the prime factors of number *a*. So, the above test gives us also, an algorithm to find the factors of a composite number.

Example 1. A prime table gives 1249 as a prime. According with our primality test, we have to check whether or not the following polynomials are factorable:

$12x^2 + 4x + 9$, $11x^2 + 14x + 9$, $10x^2 + 24x + 9$, $9x^2 + 34x + 9$, $8x^2 + 44x + 9$, $7x^2 + 54x + 9$, $6x^2 + 64x + 9$.

You can see that not one of these polynomials is factorable in terms of linear or quadratic polynomials with non negative integer coefficients. So 1243 is a prime number.

Example 2. Let us check *a* = 5041, for primality. We start with *a* itself and finished with its factors.

$5041 = 50x^2 + 4x + 1 = 49x^2 + 14x + 1 = 49x^2 + 7x + 7x + 1 = 7x(7x + 1) + (7x + 1) = (7x + 1)(7x + 1) = 71\cdot71 = 71^2$.

This shows that 5041 is composite and its prime factors are 71and 71.

The proof and its generalization will appear in the written notes of this presentation. Examples and more explanations can be seen in: (Pareja-Heredia, D. 2015).

I hope this proposal to introduce number language at elementary school, may help math education to reduce the gap between math school teaching, and the front of mathematics research, as we mentioned at ICME11 (Pareja-Heredia, D. 2008).

References

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