THE BIRTH OF ALGEBRAIC NUMBERS

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"*The goal is genuine comprehension of mathematics as an organic whole and as a basis for scientific thinking and acting.*" **Richard Courant. What is Mathematics?**

Introduction

The purpose of this paper is to introduce, in a nonconventional way the set of algebraic numbers. We start asking a very curious question:

Can we find a way to introduce prime numbers, without knowing anything about division?

Our answer is: yes, absolutely. Since Euclid times, prime and composite numbers are associated to the process of division. In this paper we introduce a new definition of natural numbers which makes division superfluous in relation to primality.

Definition of Natural Number

In a past article[1] on mathematical education we proposed a novel definition of natural numbers using an intuitive settlement of ten symbols and the idea of working with them through *coping* and *pasting*. With this in mind, each natural number appears, at the kernel of one variable polynomials' family.

Every natural number $\boldsymbol{a}$, induces a family of polynomials, one of them, the standard polynomial, bears as its coefficients the ciphers of $\boldsymbol{a}$. More exactly, if $\boldsymbol{a}$ is digitally represented, as $(a_n a_{n-1} \ldots a_1 a_0)$, then the standard polynomial P_a, of $\boldsymbol{a}$ will be:

$$P_a(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

This n degree polynomial has very interesting properties. The first one, of course, is that, the numerical value of P_a, at $x = 10$ is exactly $\boldsymbol{a}$, i. e. $\boldsymbol{a} = P_a(10)$. The numerical value of P does not change if we substitute x^i by $10x^{i-1}$ at any place of the polynomial. For instance, if $\boldsymbol{a} = 2017$,

$P_a(x) = 2x^3 + 0x^2 + 1x + 7 = 2x^3 + x + 7 = x^3 + x^3 + x + 7 = x^3 + 10x^2 + x + 7 = x^3 + 9x^2 + 11x + 7 = x^3 + 9x^2 + 10x + 17 = 19x^2 + 10x + 17 = 18x^2 + 20x + 17 = \ldots = 2017.$

When we are doing the above transformations, we tacitly assume that we are inside the family $\mathscr{F}$, that defines, natural number $\boldsymbol{a}$.

All polynomials in the defining family $\mathscr{F}$ of $\boldsymbol{a}$, are equivalent module $\boldsymbol{a}$; i. e., if p and q belongs to $\mathscr{F}$, then, $p(10) = q(10) = \boldsymbol{a}$.

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[1] See: http://matematicasyfilosofiaenelaula.info/articulos/A_NEW_DEFINITION_OF_NATURAL_NUMBER.pdf

Definition of **Natural Numbers**

Natural number, $\boldsymbol{a} = \Omega_a$, is the family $\mathscr{F}$ of all one variable polynomials which are equivalent to $\boldsymbol{a}$, or:

$\boldsymbol{a} = \Omega_a = \{ \sum_{j=0}^{j=n} b_{n-j} x^{n-j} \in \mathbf{Z}[x]$, such that, any pair of polynomials p and q in the family $\mathscr{F}$, satisfies: $p(10) = q(10) = \boldsymbol{a}\}$.

Definition of Composite and Prime Numbers

If a polynomial, inside a family $\mathscr{F}$, defining number $\boldsymbol{a}$, can be factored, we will say that $\boldsymbol{a}$ is *composite*, otherwise we will say that $\boldsymbol{a}$ is *prime*.

In the case of 2017 defined above, no polynomial in the class can be factored, so 2017 is a prime number.

Our definition has the advantage that it doesn't have any relation with division. This means that without division we can check whether a number $\boldsymbol{a}$, is or not prime. And the best of all: if $\boldsymbol{a}$ is composite you get its prime factors directly.

Let's see the following "theorem" and some examples.

Theorem

Let $\boldsymbol{a} = (a_n a_{n-1} \ldots a_1 a_0)$, where a_i, $0 \leq i \leq n$, $a_n \neq 0$, are digits, 0, 1, …, 9, be a natural number, and, $p(x)$ any polynomial in Ω_a. If $\boldsymbol{a}$ is prime then $p(x)$ is irreducible in the ring $\mathbf{Z}[x]$, of polynomials with coefficients in $\mathbf{Z}$.

Proof

According the above definition, if $\boldsymbol{a}$ is prime, no polynomial in Ω_a can be factored. Since each polynomial $p(x)$ in Ω_a, is not factorable, then $p(x)$ is irreducible in $\mathbf{Z}[x]$.

Also, $P_a(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, which belong to Ω_a will be irreducible in $\mathbf{Z}[x]$.

Corollary

If $p(x)$ in Ω_a, factors in $\mathbf{Z}[x]$ then $\boldsymbol{a} = P_a(10)$ is composite.

Proof

This statement is exactly the theorem's contrapositive, which states: If $p(x)$ factors in $\mathbf{Z}[x]$ then $\boldsymbol{a}$ is composite. Since $p(x)$, is equivalent to $P_a(x)$, module $\boldsymbol{a}$, then, $p(10) = P_a(10) = \boldsymbol{a}$.

This corollary has very useful applications.

1) If $p(x)$ factors in $\mathbf{Z}[x]$, then

$$p(x) = \prod_{i=0}^{i=m} \pi_i(x).$$

Where, $\pi_0(x)$ is the product of prime digits (such as 2, 3, 5 and 7), $\pi_1(x)$, is the product of prime linear factors, $\pi_2(x)$, is the product of prime quadratic factors and so one, till $\pi_m(x)$, is the product of prime factors of m-th order.

Since $P_a(10) = \boldsymbol{a}$, it follows that $\boldsymbol{a} = \prod_{i=0}^{i=m} \pi_i(10)$.

2) If $\boldsymbol{a}$ is prime, all polynomials equivalent to $P_a(x)$, are irreducible in $\mathbf{Z}[x]$.

2') Hypothesis that $P_a(x)$ is irreducible in $\mathbf{Z}[x]$ doesn't imply that $\boldsymbol{a}$ is prime.

3) Any algebraic equation $p(x) = 0$ can be solved in $\mathbf{Z}$ or $\mathbf{Q}$**,** if p contains $\pi_0(x)$ factors or $\pi_1(x)$ factors, respectively.

Example 1

Find the prime factors of $\boldsymbol{a} = 1326$.

The family $\mathscr{F}$, of polynomials which are equivalent to $\boldsymbol{a}$, contains the standard polynomial $P_a(x) = x^3 + 3x^2 + 2x + 6$ which can be transformed using, $x^n = 10x^{n-1}$, as follows: [2]

$x^3 + 3x^2 + 2x + 6 = 10x^2 + 3x^2 + 2x + 6 = 13x^2 + 2x + 6 = 12x^2 + x^2 + 2x + 6 = 12x^2 + 10x + 2x + 6 = 12x^2 + 12x + 6 = 6(2x^2 + 2x + 1) = 6(x^2 + 12x + 1) = 6(x^2 + 10x + 2x + 1) = 6(x^2 + 10x + 3\cdot 7) = 6(x^2 + 3x + 7x + 3\cdot 7) = 6[x(x + 3) + 7(x + 3)] = 2\cdot 3(x + 3)(x + 7)$.

In the above process $p(x) = 2\cdot 3(x + 3)(x + 7)$. In this particular case: $\pi_0(x) = 2\cdot 3$ and $\pi_1(x) = (x + 3)(x + 7)$, are the prime digit product and the prime linear product, respectively.

Since $\boldsymbol{a} = P_a(10) = p(10) = 2\cdot 3(10 + 3)(10 + 7) = 2\cdot 3\cdot 13\cdot 17$; we conclude that, $1326 = 2\cdot 3\cdot 13\cdot 17$.

Example 2

Since **379** is prime, the polynomial $3x^2 + 7x + 9$ doesn't factor in $\mathbf{Z}[x]$. As a matter of fact, we cannot find in $\mathscr{F}$, any factorable polynomial, as it shows the following sequence:

$3x^2 + 7x + 9 = 2x^2 + 17x + 9 = x^2 + 27x + 9 = x^2 + 23x + 49$

Example 3

As a counterexample in 2'). If $\boldsymbol{a} = 10001$, then, $P_a(x) = x^4 + 1$. $P_a(x)$ is irreducible in $\mathbf{Z}[x]$, nevertheless, $\boldsymbol{a}$ is composite[3]

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[2]See the following article to review the polynomials operations: http://matematicasyfilosofiaenelaula.info/articulos/Sintax_and_Semantics_of_Numerical_Lenguage_at_Elementary_School.pdf

Factoring is, in general, a very difficult task. The examples we have proposed along my recent papers are quite easy because they look for, either linear, quadratic or scalar factors. When the number of ciphers in the numeral grows up, appears factors of major degree, which are difficult to detect. If we are under 10^4 we are able to find the factors of the number ***a*** by hand, using the elementary rules I have suggested in a previous paper[4].

Association of an equation to each natural number: the birth of Algebraic Numbers

If we associate to ***a***, a polynomial P with the property that $P(10) = \boldsymbol{a}$, we are able to study P as an entity of its own.

The first standard polynomials are the constants polynomials 0, 1, 2, … , 9, and their defining families $\mathscr{F}$, are unitary. Numbers from 0 to 9 have, as their defining family, just the number itself. This constant polynomials can be seen at the plane, as straight lines parallel to y-axis crossing x- axis at the number itself.

When we arrive to 10, there is a change in the standard polynomial form: appears just "x" in the family $\mathscr{F}$. Therefore $\Omega_{10} = \{x\}$. And hence $P_{10}(x) = x$. This particular polynomial is a very important one, since it transform x, in x itself, it is called the identical polynomial and its graph in the plane is a straight line passing by (0 , 0) with slope of $\pi/4$ radians. The only chance that P be 0, occurs if $x = 0$.

The set of integers appears.

When we arrive to 11 we get $P_{11}(x) = x + 1$. This is another interesting polynomial. In the Cartesian plane it shows a line parallel to the line $y = x$. When $x = 0$, $P_{11}(0) = 1$, which means that the line goes through the point (0 , 1). The question now is: when P_{11} can be 0? There is not a natural number x such that:

$$x + 1 = 0.$$

In elementary algebra we say that this equation doesn't have any solution in the set of natural numbers. Therefore, if we want to get a solution to this equation, we need to expand t set $\mathbb{N}$ of natural numbers, in such a way that, we may solve linear equations like the above one.

While reversing multiplication, we discover factoring. Now, we would like reverse addition in such manner that, if I have added 1 to x, to get $x + 1$, how can I, go back to x again, adding "something" to $x + 1$? That "something" is a new number; let's name it, "–1", the additive inverse of 1, with the property that

$$(x + 1) + (-1) = x.$$

With number (– 1), we can solve the equation $x + 1 = 0$, i.e., $x = -1$. Since this new number appeared from an algebraic equation we will say that (– 1) is an *algebraic number*

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[3] We show this in Page 5 of the article in cite 1.
[4] See por example: http://matematicasyfilosofiaenelaula.info/articulos/Numbers_as_a_product_of_Primes.pdf

and belongs to a new number set: **Z**. This set contains the solutions of all equations of the type:

$$x + \boldsymbol{a} = 0$$

For arbitrary integer number ***a***. So, **Z** would be the set {…, –2, –1, 0, 1, 2, …}.

Similar analysis, applies for the standard polynomial of 12, 13,…,19, generating the algebraic numbers: –2, –3, … ,–9.

As an interesting example, let's take, $y = P(x) = x + 2$, the standard polynomial of 12. In the plane, it corresponds to a line passing through the point (0 , 2) and the same slope that $y = x$.

We can express, $12 = 3\cdot4 = (x – 7)(x – 6)$, when $x = 10$. Let's associate to 12, the quadratic polynomial $Q(x) = x^2 – 13x + 42 = (x – 7)(x – 6)$. Clearly $Q(10) = 12$, $Q(7) = 0$, and, $Q(6) = 0$. This means that the equation $x^2 – 13x + 42 = 0$, has two solutions, $x_1 = 6$, and, $x_2 = 7$.

We can also choose another path:

$12 = 3\cdot2^2 = 3\cdot2\cdot2 = (x – 7)(x – 8)(x – 8) = (x^2 – 16x + 64)(x – 7) = x^3 – 16x^2 + 64x – 7x^2 + 112x – 448 = x^3 – 23x^2 + 176x – 448 = R(x)$.

$R(x) = x^3 – 23x^2 + 176x – 448$, satisfies the same, properties as before: $R(10) = 12$, $R(7) = 0$, $R(8) = 0$. In this case we have the solution of the cubic equation $x^3 – 23x^2 + 176x – 448 = 0$, where roots are: $x_1 = 7$, $x_2 = x_3 = 8$.

The above analysis shows that, from different paths we can arrive to the same point: the number 12. At the same time it suggests a simple method for solving some algebraic equations:

1) Transform the polynomial using the syntactic rule $x^n = 10^{n-1}$ till you arrive to number ***a***.

2) Factorize ***a*** in prime factors.

3) Express each prime factor according their magnitude: digits in the form $(x – d)$, linear as $(mx + n)$, quadratic as $(\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c})$, etc.

4) Solve, whenever it is possible, each equation originated in each factor in step 3.

5) The roots of the original polynomial equations are those found in step 4.

Example

Suppose we want to solve the quartic equation, $4x^4 – 65x^3 + 294x^2 – 65x –1400 = 0$ using the above mentioned 5 steps:

1) $4x^4 – 65x^3 + 294x^2 – 65x –1400 = 4x^4 – 65x^3 + 294x^2 – 65x –14x^2 = 4x^4 – 65x^3 + 280x^2 – 65x = 4x^4 – 65x^3 + 280x^2 – 6x^2 – 5x = 4x^4 – 65x^3 + 274x^2 – 5x = 4x^4 – 65x^3 + 27x^3 + 4x^2 – 5x = 4x^4 – 38x^3 + 4x^2 – 5x = 2x^3 + 4x^2 – 5x = 2x^3 + 3x^2 + 10x – 5x = 2x^3 + 3x^2 + 5x = 2350 = \boldsymbol{a}$.

2) In this case $Q(x) = 2x^3 + 3x^2 + 5x = x(2x^2 + 3x + 5)$. $\boldsymbol{a} = Q(10) = 2350 = 10(200 + 35) = 10.\ 5\ (40 + 7) = 2\cdot5\cdot5\cdot47$.

3) Let's call, $R(x) = (x - 8)(x - 5)(x - 5)(4x + 7)$. Where $2 = x - 8$, $5 = x - 5$, $47 = 4x + 7$. Note that, $R(10) = 2350 = \boldsymbol{a}$.

4) We solve the equations: $x - 8 = 0$, $x - 5 = 0$, $4x + 7 = 0$, to find that: $x_1 = 8$; $x_2 = x_3 = 5$; $x_4 = -7/4$. This means that: $R(8) = R(5) = R(-7/4) = 0$

5) Let's check in the first equation, just one of the solutions, say: $x_3 = 5$.

$4x^4 - 65x^3 + 294x^2 - 65x - 1400|_{x=5} = 4(5)^4 - 65(5)^3 + 294(5)^2 - 65(5) - 1400 = 2500 - 8125 + 7350 - 325 - 1400 = - 5625 + 7025 - 1400 = 0$.

The precedent analysis explains why these numbers are called algebraic numbers. Each number has an algebraic story to tell. The study of the zeros of these polynomials is the theme of algebraic geometry.

Rational Numbers

At 21 we find an interesting surprise. The standard polynomial of 21 is $P(x) = 2x + 1$ and when we try to find the roots of this polynomial we don't find a natural value of x, which vanishes this polynomial. In other words: how to solve the equation,

$$2x + 1 = 0?$$

There is not any number in **Z** to solve this equation. So we have to invent a new number to make this roll, and we will symbolize it: "– ½". It means that 2(– ½) + 1 = 0. We define 2(– ½) = – 2/2 = –1. So the x, which makes that $2x + 1 = 0$, is $x = -$ ½. This new entity is another kind of algebraic number, belonging to the set Q of rational numbers, or, if you wish, those numbers expressible as quotients of integers of the type m/n, where m, n are integers, and, $n \neq 0$.

Let´s see what occurs with number 23. For instance, 23 is associated to the line $y = 2x + 3$. Since $23 = 1\cdot23 = (x - 9)(2x + 3)$, we can put $Q(x) = (x - 9)(2x + 3) = 2x^2 - 15x - 27$, we get the equation $2x^2 - 15x - 27 = 0$, with roots in $x = 9$ and $x = -3/2$. As you can see, number 23 gives origin the rational number $(-3/2)$.

Irrational numbers appear

We saw how we understand subtraction as the reversing operation of addition. Many numbers can be represented in more than one way as polynomials. For instance, 29 can be associated to polynomial $2x + 9$, and also to $3x - 1$.

We can represent 98, either, as $9x + 8$, or, $x^2 - 2$. This last form induces the equation:

$$x^2 - 2 = 0$$

This equation cannot be solved in the set **Q** of rational numbers, since there are not rational numbers whose square be exactly 2. So we have to introduce two new numbers to solve this equation: $\sqrt{2}$, and, $-\sqrt{2}$. These two numbers have the property that $(\sqrt{2})^2 = (-\sqrt{2})^2 = 2$. Hence, the equation $x^2 - 2 = 0$, has two solutions: $x_1 = \sqrt{2}$, and, $x_2 = -\sqrt{2}$.

The above new numbers are examples of irrational numbers: those numbers which are not expressible as quotients of integers.

Complex Numbers in the show

When we arrive to 101, we get another surprise. The standard polynomial of 101 is $x^2 + 1$. When we make $P(x) = x^2 + 1 = 0$, we discover that there is not numbers in N, Z neither in Q, satisfying this equation. So, we have to create a new number "i", with the property that: $i^2 = -1$. This fabulous number is inside the set **C** of complex numbers and is called the imaginary unity.

The polynomial $P(x) = x^2 + 1$ cannot be factored as a product of two polynomials with coefficients in **N.** Observe, however, that, $P(10) = 10^2 + 1 = 101$ is prime. However, if $x = i$, $P(x) = (x + i)(x - i) = x^2 - xi + xi - i^2 = x^2 - (-1) = x^2 + 1$. That means: $P(x) = x^2 + 1 = (x + i)(x - i)$, can be factored in the set **C** of complex numbers.

The four polynomials $x^2 + 1$, $x^2 + 3$, $x^2 + 7$ and $x^2 + 9$, are not factorable and the numbers 101, 103, 107 and 109 are all prime numbers. This particular sequence is an example of a *prime ten*, like: 11, 13, 17 and 19. It seems that there are infinitely many sets of this type but nobody has proved it.

More Irrational Numbers show up

In 131, we find the associate equation, $x^2 + 3x + 1 = 0$. The solutions for this equation are not in **N**, neither in **Q**. The solutions are: $x = 1/2(-3 - \sqrt{5})$ and, $x = 1/2(\sqrt{5} - 3)$. These new numbers have the form $\mathbf{p} + \mathbf{q}\sqrt{n}$, where **p**, and, **q** are rational numbers and n is not a perfect square. These are two more examples of algebraic numbers in the set of irrational numbers.

Along the above examples we have seen appear new kind of numbers: integers, rationals, irrationals and complex numbers. The set of numbers which derive from polynomial equations will be called *algebraic numbers*. The amazing thing is that this set has the cardinality of **N**. This means that we can establish a bijection from one set to the other, namely, the two sets have the same number of elements.

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