ROOTS AND STEMS

Number Theory for Kids

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"*Martin* (*Gardner*) *has turned thousands of children into mathematicians and thousand of mathematicians into children*". Ronald L. Graham

Abstract
This paper is dedicated to the memory of Martin Gardner (1914-2010) who taught to all of us that in mathematics, one way to reach higher limits is beginning with the curiosity, innocence and ingenuity of children. In this paper we propose to make a number theory based in a new definition of natural numbers, where addition and multiplication are inside the kernel, from which, prime and composite numbers are born.

Introduction

This paper aspires to be another echelon in the improvement of basic arithmetic teaching for kids, aimed to teachers of elementary school mathematics. Its central claim has to do with the proposal of changing the definition of numbers.[1] In this way, we could make number theory more comprehensible to kids. Some articles, where to find examples and a longer treatment of the different topics studied here, appear cited at foot notes.[2]

We start with colloquial language and throughout the use of natural routines of adding (collecting) and multiplying (repeating) we can reach numerical language. Colloquial language structure can help us to make sense of numbers words. When we say "*one thousand nine hundred fourteen*" or "*nineteen hundred fourteen*" we are meaning: one time thousand, and, nine times hundred, and, one ten, and, four, or, if you prefer, nineteen times hundred, and, fourteen. Here appear two key words with a mathematical meaning, namely: "and", "times". These are the colloquial words to signify addition and multiplication.

The Kernel inside the origin of Natural Numbers

We can imagine numbers as seeds, with two gametes inside: addition and multiplication. These two operations, through simple syntax rules, together with some nutrients called *digits* can give origin to all natural numbers. Digits are the basic and most simple elements to built numbers. In the case of our decimal numerical system these digits are in the set {0,1,2,3,4,5,6,7,8,9}. The most simple, of all positional systems, is called binary system with just two digits: 0 and 1.[3]

To begin with, let us accept that any number can be represented as a sum of products of a digit times a power of ten, in an order previously established. For example the number 2314 can be represented by:

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[1] From here on, we use the word number to mean natural number, at least we mean otherwise.
[2] Most of these articles are at: http://www.matematicasyfilosofiaenelaula.info/
[3] See: *The power of two* at:
http://www.matematicasyfilosofiaenelaula.info/conferencias/El%20poder%20del%20Dos2.pdf

$2314 = 2000 + 300 + 10 + 4 = 2·10·10·10 + 3·10·10 + 1·10 + 4 = 2·10^3 + 3·10^2 + 1·10 + 4 = 2·x^3 + 3·x^2 + 1·x^1 + 4·x^0 = 2x^3 + 3x^2 + x + 4$, with x representing 10.

In the final steps above, we changed 10 by x, to emphasize the exponents' regularity and to show the appearance of a common algebraic expression: a polynomial. As we have shown in other works it is possible to change numbers by polynomials to perform sums and multiplications inside them.[4]

Definition 1. In a general footing, if the number a, is represented with digits, $a_n a_{n-1} \ldots a_1 a_0$, in that order, the numerical value of $a = (a_n a_{n-1} \ldots a_1 a_0)$, can be expressed by

$$a = (a_n a_{n-1} \ldots a_1 a_0) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{i=n} a_{n-i} x^{n-i}.$$

When we refer to a, written in the digital form $a_n a_{n-1} \ldots a_1 a_0$, we will call it, a *scalar*.

Root and Stem of a Number

Any natural number a, has two parts: a **root** and a **stem**. The root is built by factoring and the stem is the addition part at the end of the number representation, namely, the last digit. Above: the root of a is always a multiple of ten and it is given by $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x$ and its stem is just a_0. So any number can be expressed as a sum of *linear* type:
$a = kx + l = (a_n a_{n-1} \ldots a_1)x + a_0$, where k is a scalar and l is a digit from $\{0,1,2,3,4,5,6,7,8,9\}$.

For instance the root of 1916 is $1910 = 191x$ and its stem is 6, so $1916 = 191x + 6$, corresponding to an arrange of a scalar times x plus a digit.

Inside any number, we find a root and a stem. Digits are stems by themselves with root zero. There is a special characteristic in the root: it is a *composite number*, namely, is a product of at least, two numbers different from 1. On the other hand, it is adding of stem that gives numbers their quality of being, either *prime* or *composite*.

The stem gives you the first information about numbers. For example, if the stem is 0, 2, 4, 6, or, 8, the number is a multiple of 2, and so it is an even number, otherwise the number should be odd.

If the stem is zero ($a_0 = 0$), the number a, has the form:

$$a = (a_n a_{n-1} \ldots a_1) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = (a_n x^{n-1} + a_{n-1} x^{n-2} + \ldots + a_1)x = 2·5(a_n 10^{n-1} + a_{n-1} 10^{n-2} + \ldots + a_1) = 2·5·(\sum_{i=1}^{i=n} a_{n-i+1} 10^{n-i})$$

And so, the number is at least, a multiple of two and five. In this case a can be written as:

$$a = kx, \text{ where } k = a_n x^{n-1} + a_{n-1} x^{n-2} + \ldots + a_1$$

[4] All this techniques are described at:
http://www.matematicasyfilosofiaenelaula.info/articulos/Sintax_and_Semantics_of_Numerical_Lenguage_at_Elementary_School.pdf

As you could observe, numbers are, either even or odd, according their stem be either even or odd, without any additional consideration.

In other paper[5] we introduce the definition of natural number as an equivalence class of congruent polynomials module a, with the syntactic property

$$x^n = 10x^{n-1} \qquad (*)$$

The meaning of this expression is: the value of n-th position in the numeral is ten times the value of (n-1)-th position, counted from 0 to n and from right to left.

For example the number 1916 can be associated to the polynomial $1\cdot x^3 + 9\cdot x^2 + 1\cdot x + 6$. By the syntactic property (*), $1\cdot x^3 = 1\cdot x\cdot x^2 = 10\cdot x^2$ and replacing, we find $1\cdot x^3 + 9\cdot x^2 + 1\cdot x + 6 = 10\cdot x^2 + 9\cdot x^2 + 1\cdot x + 6 = 19\cdot x^2 + 1\cdot x + 6 = 19\cdot 10^2 + 1\cdot 10 + 6 = 19\cdot 100 + 16$. That's the reason why we say, *nineteen hundred sixteen* instead of one thousand nine hundred sixteen.

Addition and multiplication using roots and stems

Let, $a = r_1 + s_1$ and $b = r_2 + s_2$, two numbers represented with their root and stem parts, then we define addition, $a + b$, and multiplication, $a \cdot b$, of these two numbers, in the following terms:

Addition: $a + b = (r_1x + s_1) + (r_2x + s_2) = (s_1 + s_2) + (r_1x + r_2x) = (r_1 + r_2)x + (s_1 + s_2)$.

Example. If the numbers are $a = 29$, and, $b = 75$, their roots and stems are respectively: $r_1 = 20 = 2x$, $s_1 = 9$; $r_2 = 70 = 7x$, $s_2 = 5$, and so,

$a + b = 29 + 75 = (2x + 9) + (7x + 5) = (9 + 5) + (7x + 2x) = 14 + 9x = 4 + x + 9x = 10x + 4 = 104$

The adding process above can be simplified through the following schema

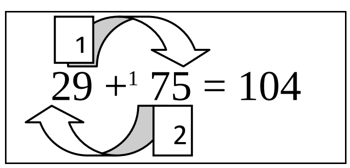

***Figure* 1**. **The adding schema.** We start the process following the upper arrow, adding 9 plus 5 equal to 14. We write 4 at right and keep a carry of 1 (written at the middle). The low arrow, takes the 1 carry, and adds to 7 + 2, to get 1 + 7 + 2 = 10. These two digits are placed aside 4 to get the final sum 104. Look at the arrows orientation: both are clockwise.

[5] Definition and its consequences are given at: http://www.matematicasyfilosofiaenelaula.info/articulos/A_Genesis_of_Natural_Numbers.pdf

Multiplication by scalar[6]. Let's take the case when the first factor is a scalar, say *k*, and the second factor is an ordinary number of the form $(r_1 + s_1)$, then the product is given by

$$k \cdot (r_1 + s_1) = k \cdot r_1 + k \cdot s_1$$

Suppose we have $k = 7$ and $(r_1 + s_1) = 20 + 9$, then, the product is:

$$7 \cdot (20 + 9) = 7 \cdot 20 + 7 \cdot 9 = 140 + 63 = 203$$

General Multiplication. In the general case the process is a little longer.

$$ab = a \cdot b = (r_1 + s_1) \cdot (r_2 + s_2) = (s_1 \cdot s_2) + (s_2 \cdot r_1) + (s_1 \cdot r_2) + (r_2 \cdot r_1)$$

Let's take again $a = 29$, and, $b = 75$. Then:
$a{\cdot}b = 29{\cdot}75 = (20 + 9){\cdot}(70 + 5) = (9{\cdot}5) + (5{\cdot}20) + (9{\cdot}70) + (70{\cdot}20) = (14{\cdot}100) + (9{\cdot}7{\cdot}10) + (5{\cdot}2{\cdot}10) + (9{\cdot}5) = 45 + 100 + 630 + 1400 = 1400 + 500 + 130 + 100 + 45 = 2175$.

The above multiplication process follows, let's say, a linear way, without getting out of the context. We start with an input and go through a sequence of logical steps till you get, at the end, an output. This procedure can be reduced to another simple schema as shown in figure 2.

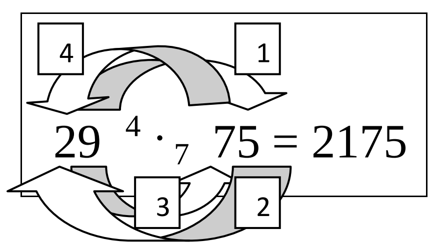

***Figure* 2. The multiplication schema**. Above appears a diagram for the product 29·75. The first arrow multiplies 9 by 5 to get 45. We write 5 at the right side of the equal sign and we have a carry of 4 (written at the middle). The second arrow makes the multiplication, 5 times 2 and adds the carry 4, to get 14. The third arrow adds 14 to the product 9·7 = 63, to get 77. We write 7 aside 5 and get a carry of 7 (written at the middle). Fourth arrow adds the carry 7 to the product 7·2 = 14, getting, 7 + 14 = 21. This number is written aside 75 to get 2175 as the product of 29 times 75. The first two arrows are clockwise, the following two, are counterclockwise.

[6] The term scalar is associated to the *scaling* process. When $k > 1$, the product grows, when $0 < k < 1$, the product contracts and if $k < 0$, the product changes its sign.

We can extend easily the above processes for factors with more than two digits, just following definition with roots and stems. For instance, $123\cdot 456 = (120 + 3)(450 + 6) = 3\cdot 6 + 6\cdot 120 + 3\cdot 450 + 450\cdot 120 = (45)(12)(100) + 3\cdot 45\cdot 10 + 6\cdot 12\cdot 10 + 18 = 18 + 720 + 1350 + 54000 = 56088$.

A General Definition of Natural Numbers[7]

Definition 2

Two polynomials $p(x)$ and $q(x)$ associated to a number a, are said equivalent module a, symbolically, $p(x) \equiv q(x) \pmod{a}$, whenever, $p(10) = q(10) = a$. Here $p(10)$ and $q(10)$ means the numerical value of p and q whenever $x = 10$.

Example

The number 2016 has a polynomial standard representation as: $p(x) = 2\cdot x^3 + 0\cdot x^2 + 1\cdot x + 6 = 2x^3 + x + 6$. Another polynomial associated to 2016 is $q(x) = 20x^2 + 16$. In this case we say that $2x^3 + x + 6$, and, $20x^2 + 16$ are equivalent module 2016. Symbolically, $2x^3 + x + 6 \equiv 20x^2 + 16 \pmod{2016}$

Definition 3 (A new definition for Natural Number)

We define number $a = (a_n a_{n\text{-}1}\ldots a_1 a_0)$, *as the family* Ω_a *of all equivalent polynomials module* a, namely:

$$\Omega_a = \{ \sum_{j=0}^{j=n} b_{n-j} x^{n-j} \text{, such that, any pair of polynomials } p, q, \text{ in the family satisfies } p(10) = q(10) = a \}.$$

[a_i, $0 \le i \le n$, are digits written in the usual order].

In this family are enclosed *all* polynomials p, such that $p(10) = a$. If we work inside set Ω_a we may see all polynomials as equal and there will be no ambiguity to replace one of them by another, whenever we need it. We can also identify Ω_a with a.

So, we can say for example, that number 348 can be defined, in a restrictive way, by just three polynomials of different degree since all of them give 348 when $x = 10$.

$$348 = \{3x^2 + 4x + 8, 34x + 8, 348\}.$$

Definition 4 (An alternative definition of Prime Number)

A number $\boldsymbol{a}$, *is prime if and only if, inside* $\Omega_{\boldsymbol{a}}$ *there are not factorable polynomials. Otherwise* $\boldsymbol{a}$ *is composite*.

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[7] Examples and more information relating this topic can be found at: http://www.matematicasyfilosofiaenelaula.info/conferencias/The_Number_Language_eimat2015_Conferencia.pdf

If we want to test a number for primality we could check first its stem. If a number (different of 2 or 5) has its stem: 0, 2, 4, 5, 6 or 8, the number is composite. That is because, if a_0 is one of the given digits, then we can get a common factor inside a.

Example: The number 3645 is composite because its stem is 5. In fact:

$3645 = 3x^3 + 6x^2 + 4x + 5 = r + s = (3x^3 + 6x^2 + 4x) + 5 = (3x^2 + 6x + 4)\, x + 5 = (3x^2 + 6x + 4)\, 2\cdot 5 + 5 = 5[2(3x^2 + 6x + 4) + 1] = 5\cdot 729$.

This shows that 3645 is composite.

Let k a scalar and $a = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then:

1) $k\cdot(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0) = k\cdot a_n x^n + k\cdot a_{n-1}x^{n-1} + \ldots + k\cdot a_1 x + k\cdot a_0$.

Since equality is a symmetric relation we can also get the following:

2) $k\cdot a_n x^n + k\cdot a_{n-1}x^{n-1} + \ldots + k\cdot a_1 x + k\cdot a_0 = k\cdot(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)$

In 1) we are getting a product and in 2) we are factoring. This last process is quite important for the remaining of this paper, since we want to establish some useful tests of primality.

A set of factoring examples

When we say that 61 is a factor of 6527, we are really meaning that, there exist a polynomial $p(x)$ such that, $61\cdot p(x) = 6527$. In this case 61 is a scalar and we have to find $p(x)$.

To find $p(x)$ we start with the standard representation of 6575 and through simple transformations, based on the syntactic property, we arrive to $p(x)$.

$6527 = 6x^3 + 5x^2 + 2x + 7 = 65x^2 + 2x + 7 = 61x^2 + 4x^2 + 2x + 7 = 61x^2 + 42x + 7 = 61x^2 + 7\cdot 6x + 7 = 61x^2 + 7\cdot(6x + 1) = 61x^2 + 7\cdot 61 = 61\cdot(x^2 + 7)$. From here we conclude that $p(x) = x^2 + 7$. Easily we can verify that $61\cdot 107 = 6527$.

If $m = p\cdot q$, then there exist a polynomial $q(x)$, such that, $m = p\cdot q(x)$. Similarly, there exist a polynomial $p(x)$, such that, $m = q\cdot p\,(x)$. Let's verify this fact in the above example, this time with factor 107.

$6527 = 6x^3 + 5x^2 + 2x + 7 = 65x^2 + 2x + 7 = 60x^2 + 52x + 7 = 6\cdot 10x^2 + 10x + 42x + 7 = 10x(6x + 1) + 7(6x + 1) = (10x + 7)(6x + 1) = (x^2 + 7)(6x + 1) = 107(6x + 1) = 107\cdot q(x)$. Here, $q(x) = 6x + 1$.

In a general context, if m factors as: $m = p_1 p_2 \ldots p_r$, where $p_1, p_2, \ldots, p_r$, are primes, then for each prime factor, there exist a polynomial q, such that $m = p\cdot q(x)$.

In some cases we are interested in knowing all the factors of a composite number. Let's take, as an example, the composite number 6300 and try to find all its prime factors.

Number 6300 is really, a very interesting number: it is just in the middle of two twin prime numbers, 6299 and 6301. It also has the first four prime factors in sequence. I call this kind of numbers, *trusting numbers of first class*. Some examples are: $30 = 2 \cdot 3 \cdot 5$; $420 = 2^2 \cdot 3 \cdot 5 \cdot 7$; $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ and $11550 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$.

$6300 = 6x^3 + 3x^2 \ ^{\text{(Factoring } x \text{ square)}} = x^2(6x + 3) \ ^{(x = 2.5)} = 2^2 \cdot 5^2(6x + 3) \ ^{\text{(Taking 3 out)}} = 2^2 \cdot 3 \cdot 5^2(2x + 1) \ ^{(2x + 1 = 3.7)} = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7$. So 6300 is a trusting number of first class.

If we were in capacity of showing that there exists infinity trusting numbers of first class, we could say that the twin primes conjecture would be proved.

Multiplication and a reverse procedure

Reversing a mathematical process not always is a simple task. For the case, of $ab = m$, we can recover a and b knowing m, whenever we know that a and b are prime numbers.

We learned that multiplication is a cyclic procedure like this:

$$(r_1 + s_1) \cdot (r_2 + s_2) = (s_1 \cdot s_2) + (s_2 \cdot r_1) + (s_1 \cdot r_2) + (r_2 \cdot r_1) \qquad (**)$$

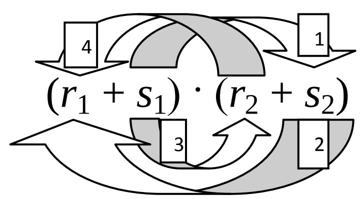

***Figure* 3**. Here we have a schematic representation of the product of two numbers in their typical form through roots and stems. When the two factors are primes we can reverse the process to recover the factors.

And now the question is: how this process can be reversed, when we know the right side of (**)? We can do it, of course, but it is not always, an easy task. The problem is that right side of (**) usually is given as a number m and we need to find roots and stems of the original factors. Moreover the problem might have two or more answers. Think, for instance, that you have been asked from where, 224 comes from through multiplication. You can answer: 4 and 56, 8 and 28, 16 and 14 and a couple of more pairs. So the answer is not unique. However, if the question is related to prime decomposition, the answer is really unique. For that case we would be interested to find primes numbers, such that, their product be m.

In the case $m = 224$, prime factors can be found through the standard polynomial, $2x^2 + 2x + 4$.

$2x^2 + 2x + 4 = 2(x^2 + x + 2) = 2^2(5x + 6) = 2^3(2x + 8) = 2^4(x + 4) = 2^5 \cdot 7$.

From $2^5 \cdot 7$, we can find all possible factors for 224, among them: 32 and 7, 16 and 14, 8 and 28, etc.

We are now in capacity to factorize numbers large enough for elementary school. Let us take for instance 6731.

$6731 = 6x^3 + 7x^2 + 3x + 1 = 67x^2 + 3x + 1 = 65x^2 + 23x + 1 = 64x^2 + 33x + 1 = 62x^2 + 53x + 1 = 60x^2 + 73x + 1 = 60x^2 + 71x + 7·3 = 60x^2 + 36x + 35x + 7·3 = 12x(5x + 3) + 7(5x + 3) = (5x +3)(12x + 7) = (5x + 3)(x^2 + 2x + 7) = 53·127$.

As you can observe in this example, the stem (1), of the product, comes from the stem of 21 = 3·7, the product of stems of the two factors.

The following is an example of a number with three linear factors.

$6919 = 6x^3 + 9x^2 + x + 9 = 69x^2 + x + 9 = 66x^2 + 31 x + 9 = 66x^2 + 22x + 99 = 11(6x^2 + 2x + 9) = 11(4x^2 + 22x + 9) = 11(4x^2 + 18x + 7·7) = 11(3x^2 + 28x + 7·7) = 11(3x^2 + 21x + 7x + 7·7) = 11[3x(x + 7) + 7(x +7)] = (x +1)(x + 7)(3x +7) = 11·17·37$.

Linear factors have the form $ax + b$, (b, a digit). However a, could be any scalar, and so, **it is possible to extend the above procedure to any polynomial representing a natural number**. We have first, of course, to put all digital prime factors, namely 2, 3, 5 and 7, out of the polynomial.

For example, to find prime factors of 27889, we have to look for r_1, r_2, s_1 and s_2 such that: $(r_1 + s_1) \cdot (r_2 + s_2) = (s_1 \cdot s_2) + (s_2 \cdot r_1) + (s_1 \cdot r_2) + (r_2 \cdot r_1) = 27889$.

The stem of 27889 is 9. This stem comes from the stem of $(s_1 \cdot s_2)$. The possibilities are: either, both s_1 and s_2 are equal to 3 or 7 (their product is 9 or 49), either s_1, or s_2 are 9, and the other one is 1. On the other hand, $27889 = 278x^2 + 8x + 9$ looks like a perfect square and with a little work we get:

$27889 = 278x^2 + 8x + 9 = 256x^2 + 22x^2 + 8x + 9 = 256x^2 + 220x + 8x + 9 = 256x^2 + 228x + 9 = 256x^2 + 224x + 49 = (16x)^2 + 2·16·7x + 7^2 = (16x + 7)^2 = 167^2$.

Then $r_1 = r_2 = 16$, and $s_2 = s_1 = 7$ and $27889 = (r + s)^2 =(16 + 7)^2$.

A General Test of Primality

If $a = (a_n a_{n-1} \dots a_1 a_0)$, then a is prime whenever in the set of quadratic equivalent polynomials module a,

$$\{(a_n a_{n-1} \dots a_2)x^2 + a_1 x + a_0, ((a_n a_{n-1} \dots a_2)-1)x^2 + b_1 x + b_0 \dots, [(a_n a_{n-1} \dots a_2)/2]x^2 + u_1 x + u_0\}$$

There are not factorable polynomials. Here $[m]$ means the greatest integer less or equal to m.

Let us check for primality number $a = 1627$. The family to be tested is

$\{16x^2 + 2x + 7, 15x^2 + 12x + 7, 14x^2 + 22x + 7, 13x^2 + 32x + 7, 12x^2 + 42x + 7, 11x^2 + 52x + 7, 10x^2 + 62x + 7, 9x^2 + 72x + 7, 8x^2 + 82x + 7\}$.

The first polynomial, $16x^2 + 2x + 7$, is not factorable since there are not two numbers k and l such that $k \cdot l = 16 \cdot 7$, and, $k + l = 2$. Same argument can be applied to the other polynomials. We do not check further since in the last polynomial $8x^2 + 82x + 7$, $82 > 8 \cdot 7$ and from there on, the product of the coefficient of x^2 times the independent term will be always less than the coefficient of x.

If we test $a = 1629$, we find that, the first polynomial, $16x^2 + 2x + 9$, in the family is equivalent module a to:

$$16x^2 + 2x + 9 = 15x^2 + 12x + 9 = 3(5x^2 + 4x + 3) = 3(3x^2 + 24x + 3) = 3^2(x^2 + 8x + 1).$$

Since 181 is prime, we conclude that, $1629 = 3^2(x^2 + 8x + 1) = 3^2 \cdot 181$, is composite.

Noteworthy is the fact that, in using roots and stems, all composite numbers can be written as a product of linear factors of the type $ax + b$, with scalar a and a digit b.

Conclusion

Since natural numbers can be represented as a sum of roots and stems, we can do a new type of arithmetic teaching. Particularly addition and multiplication are quite easy to learn and remember. Kids also can learn factoring in a reasonably way without knowing division. We expect not to spend so much time in elementary arithmetic but teaching a little more elementary number theory.

Armenia, Colombia, February 2016