A NEW DEFINITION OF NATURAL NUMBERS
And its Incidence on Teaching Mathematics

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"To *go wrong in one's own way is better than to go right in someone else's*."
— Fyodor Dostoyevsky, **Crime and Punishment**

Introduction

The purpose of this paper is to introduce at elementary school the concept of natural number using very intuitive ideas: copying and pasting. The starting point is a set of ten basic symbols and the analogy of addition and multiplication with pasting and copying. Most of the material here is elementary and it is thought for arithmetic teachers interested to see mathematics in a quite different way than that in the traditional curriculum.

The philosophy that animates this article has its origin in Gottlog Frege[1], who was the first mathematician to understand the importance of the concept of definition in mathematics, when he said: "is above all, number, which must be defined or recognized as indefinable." His interest in defining number is related with proving the arithmetical laws. He introduces the concept of ancestry in order to define the concept of natural number[2].

Why a new definition could be useful? In our case, at least because, a new definition for natural numbers let us to give another view of prime numbers without consider the traditional way of use division. We use factoring instead. We can also understand arithmetic algorithms in a reasonable way and do not through memorizing boring routines.

Our elementary approach combines arithmetic and algebra; from basic operations to solutions of equations, including equations of degree greater than two, when some of their solutions are in **Q,** the set of rational numbers. Using our methodology, it is possible to introduce early in teaching mathematics, notions from linear algebra and advanced mathematics. This approach let us introduce at elementary school, numbers sets like **Z**, **Q** and **R,** including the set of algebraic numbers and concepts like irrationality, countability and uncountability.

This is not a technical paper with a long list of definitions, lemmas and theorems. Instead I have chosen the way of introducing all material through examples and motivations to go further in studying more deep ideas.

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[1] See the interesting paper: WAGNER, S. *Frege's Definition of Number*
Notre Dame Journal of Formal Logic. Volume 24, Number 1, January 1983. Available at the web

[2] See my epistemology notes in:
http://matematicasyfilosofiaenelaula.info/Epistemologia%202009/Los%20Números%20Naturales%20y%20el%20Concepto%20de%20Buena%20Ordenacion.pdf

Equivalence Classes

Dividing math objects in classes is a common methodology. For instance, natural numbers can be seen as the union of two disjoint sets: odd and even numbers. Even numbers are multiples of 2, while odd numbers are not.

We are interested here, in some special classes $\mathscr{F}$, namely, classes with the property that, whenever we have elements *a* ,*b*, *c*, *d*, …, in the class and a relation *R* defined in the class, it must satisfy:

Reflexivity: *aRa*
Symmetry: *aRb*, implies, *bRa*.
Transitivity: If *aRb*, and, *bRc*, then, *aRc*.

If *R* satisfies the above properties in a given set, this set is partitioned by *R* in disjoint sets, in such way that their union reproduces the initial set.

The set $\mathbb{N}$ of natural numbers {0,1, 2,…} has being defined in many ways through the last millennia. Here we propose a new constructive definition trying to arrive to a different way to see natural numbers and their properties. To begin with, let us use an intuitive approach, now very used in the computer age: copying and pasting. Copying in our context means repeating a process or action; in the numerical case, this means, multiplication. Similarly, pasting means put together a pair of objects and in the numeric case, it means addition.

Let's take for granted the existence of a basic set of symbols to represent graphically the set $\mathbb{N}$, of natural numbers, say: D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}[3]. We also assume that it is possible to define two inner operations: +, and, $\cdot$ in $\mathbb{N}$, in such way that if $a, b \in \mathbb{N}$ then, $a + b \in \mathbb{N}$ and, $a \cdot b \in \mathbb{N}$.

Then, pasting *a*, *b* in $\mathbb{N}$ will mean $a + b$ and copying *a*, *b* in $\mathbb{N}$, should mean, take *a* copies of *b*, or *b* copies of *a*, in symbols, $a \cdot b$ or simply, *ab*. For example in D, we can mean, paste 2 and 3, as 2 + 3 and copy 2 and 3, as, $2 \cdot 3$, or $3 \cdot 2$. Copying *a*, *a*, *a*, *a*, …, will be expressed as: a^2, a^3, a^4, …, respectively.

Representing $\mathbb{N}$ as families of some special polynomials

The set D, defined above, is called a base for representing natural numbers. This set has ten elements and a representation of natural numbers using this base is called a decimal representation of $\mathbb{N}$. The origin of this representation goes back to ancient India, and was the Arabs who brought this decimal system to west around middle Ages. Let us call *x*, the number of elements of D, namely, the cardinal of D, in this case, $x = 10$. The powers of ten, 10^0, 10^1, 10^2, 10^3, ... , will be represented by 1, x, x^2, x^3, …. Note that for the case of our decimal positional system, $x^n = 10x^{n-1}$. This means that, the relative *n*-th position in the

[3] We can also select different bases as short as {0, 1} to represent all natural numbers.

numeral is the same than 10 times the (n–1) position. This syntactic property let us represent natural numbers in several forms using polynomials.

The sequence of natural numbers using copying and pasting appears as: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, x, $x + 1$, …, $x + 9$, $2x$, $2x + 1$, …, $2x + 9$, $3x$, $3x + 1$, …, $9x + 9$, x^2, $x^2 + 1$, …, $2x^2$, $2x^2 + 1$,…, $9x^2 + 9$, …, $9x^2 + 9x + 9$, x^3, $x^3 + 1$, … .

Number 2016, using copying and pasting looks like: $2x^3 + x + 6$. Forgetting that $x = 10$, an expression as, $2x^3 + x + 6$, looks like an algebraic object called one variable polynomial. In other papers[4] I have proposed to perform arithmetic operations, using numbers expressed as polynomials.

Definition of Standard Polynomials

Let $\mathbb{N}[x]$ be the set of all one variable polynomials with coefficients in $\mathbb{N}$. Let a be a natural number written in its standard decimal representation,

$a = (a_n a_{n\text{-}1} \ldots a_1 a_0) = a_n\, x^n + a_{n\text{-}1} x^{n\text{-}1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{i=n} a_{n-i} x^{n-i}$, with $x = 10$ and $a_i \in \{0, 1, 2, ... , 9\}$.

We call polynomial $\mathrm{P}_a(x) = \sum_{i=0}^{i=n} a_{n-i} x^{n-i}$, the *standard polynomial* associated to a.

This means that a natural number is uniquely associated to a standard polynomial and vice versa any standard polynomial is uniquely associated to a natural number. For instance, the standard polynomial associated to 23578 is $2x^4 + 3x^3 + 5x^2 + 7x + 8$. Also a polynomial like $\mathrm{P}(x) = 2x^4 + 3x^3 + 5x^2 + 7x + 8$, is associated to the natural number 23578.

$\mathbb{N}[x]$ is much more than the class of standard polynomials, since polynomials like $12x^3 + 28x^2 + 2x + 43$ belongs to $\mathbb{N}[x]$, but it is not a standard polynomial. For each natural number a we can also associate to it, all polynomials $\mathrm{P}(x)$ such that $\mathrm{P}(10) = a$

The number $a = 124$ is associated to its standard polynomial $\mathrm{P}(x) = x^2 + 2x + 4$. However we can also associate to it, polynomials like, $x^2 + 24$, $12x + 4$, even, 124, since, when we replace x for 10 we get 124.

In a general context, to a number a, we can associate all a family of polynomials of a variable x. More exactly, all polynomials P, such that, $\mathrm{P}(10) = a$. These classes $\mathscr{F}_a$ are equivalent classes inside $\mathbb{N}[x]$, that is, if f and g, belong to $\mathscr{F}_a$ then $f(10) = g(10) = a$. We say that f and g are **congruent module** a. When we previously know that f and g are in $\mathscr{F}_a$, we will say that $f = g$.

Definition of Congruent Polynomials

We say that two polynomials P and Q are congruent module a (or equivalent module a), whenever $\mathrm{P}(10) = \mathrm{Q}(10) = a$. A pair of polynomials may be congruent but not necessarily

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[4] See for example:
http://matematicasyfilosofiaenelaula.info/articulos/Beginning_Abstract_Algebra_at_Elementary_School.pdf and http://matematicasyfilosofiaenelaula.info/articulos/ROOTS%20AND%20STEMS%20V.1.pdf

equal. It is the case when, P(x) = $x^2 + 2x + 4$ and Q(x) = $12x + 4$. P and Q are congruent module 124, but they are not equal in $\mathbb{N}[x]$; the first one is a quadratic polynomial, the second one is a linear polynomial, indeed, one is different from the other. When we are inside an equivalent class we consider the elements on it, as equals. I hope this ambiguity does not create confusion.

Definition of natural numbers using equivalent classes

We define number $a = (a_n a_{n-1} \dots a_1 a_0)$, as the family $\mathscr{F}$, of all equivalent polynomials module a. Symbolically

$\Omega_a = \{ \sum_{j=0}^{j=n} b_{n-j} x^{n-j} \in \mathbb{N}[x]$, such that, any pair P and Q of polynomials in the family $\mathscr{F}$, satisfies: P(10) = Q(10) = a}.

According the above definition we can identify the number a with Ω_a, the class of all polynomials congruent to a.

When we are inside Ω_{124}, P(x) = $x^2 + 2x + 4$ and Q(x) = $12x + 4$ are congruent module 124, or equivalent module 124, since P(10) = Q(10) = 124. When there is not risk of ambiguity we usually say: $x^2 + 2x + 4 = 12x + 4$, because of, our assumption that P(10) = Q(10) = 124.

The above definition let us introduce prime numbers without making any allusion to division. The reason is that through polynomial usage, the inverse process to multiplication is factoring. This means that, multiplication of polynomials in Ω_a may be reverse using factoring. Suppose that a = 2263. If P(x) = $3x + 1$ and Q(x) = $7x + 3$, the product, P(x)Q(x) = $(3x + 1)(7x + 3) = 21x^2 + 7x + 9x + 3 = 21x^2 + 16x + 3$. In Ω_a, $21x^2 + 16x + 3 = 22x^2 + 6x + 3 = 2x^3 + 2x^2 + 6x + 3$ = 2263. This last product can be factored reversing the process to recover P(x) and Q(x) and so 2263 = P(10)Q(10) = 31·73. ([5]) If a number a is associated to a standard polynomial $S(x)$ = P(x)Q(x) in Ω_a, then a = P(10)Q(10) is a composite number. From here we can get the following definition.

Definition of Composite and Prime Numbers

A number a, defined using Ω_a, is said to be *composite* if there exists a polynomial P in Ω_a, which can be factored. Otherwise a will be called *prime*.

Example

The number 747 = Ω_{747} = {$7x^2 + 4x + 7$, $6x^2 + 14x + 7$, $6x^2 + 12x + 27$, … , 747}. Since the polynomial $6x^2 + 12x + 27$ can be factor as $3(2x^2 + 4x + 9)$, we can say that 747 is composite. Moreover, 249 = $2x^2 + 4x + 9 = 24x + 9 = 3(8x + 3)$; and so, 747 = $3(2x^2 + 4x +$

[5] See my work: *How to Express Counting Numbers as a Product of Primes. Beginning Abstract Algebra at Elementary School.* http://matematicasyfilosofiaenelaula.info/articulos.htm

9) = 3[3(8x + 3)] = $3^2(8x + 3) = 3^2 \cdot 83$. Since 3 and 83 are prime numbers, we have found the representation of 747 as a product of primes without using division at all.

We understand by factoring the inverse process of multiplication, in other words, this is the process of recovering a and b, whenever you know that $c = a \cdot b$, with, $a > 1$ and $b > 1$. For instance, if you previously know that 1633 is the product of two factors a and b, then you may recover those factors, just factoring 1633 as a product of two factors; indeed, inside the class $\Omega_{1633,}$ we can establish:

$1633 = x^3 + 6x^2 + 3x + 3 = 10x^2 + 6x^2 + 3x + 3 = 16x^2 + 3x + 3 = 15x^2 + 13x + 3 = 14x^2 + 23x + 3 = 14x^2 + 21x + 2x + 3 = 7x(2x + 3) + (2x + 3) = (7x + 1)(2x + 3)$.

So you can choose $a = 7x + 1 = 71$ and $b = 2x + 3 = 23$. Therefore $1633 = 71 \cdot 23$.

An example taken from a recent article[6] asks for factors of the polynomial, $f(x) = x^8 + x^7 + x^5 + x^3 + x - 1$ in **Z**[x]. We can associate this polynomial to the number 110.101.009, assuming that $x = 10$. The problem here would be factoring this number first, and then, try to discover through its factors, the polynomials associated to these numbers.

When I ask the Wolfram app in my phone for the prime factors of this number, I get: 73, 101, 109, and 137. These numbers can be associated to the polynomials $7x + 3$, $x^2 + 1$, $x^2 + 9$, $x^2 + 3x + 7$. However, the authors of the paper found, $x^2 + 1$, $x^2 + x - 1$ and $x^4 + 1$, as irreducible factors of $f(x)$. My explanation is that, since $x = 10$, $x^2 + x - 1 = x^2 + 9$; also using the syntactic rule, $x^n = 10x^{n-1}$, the product $(7x + 3)(x^2 + 3x + 7) = x^4 + 1$. The polynomial $x^4 + 1$ is irreducible in the ring **Z**[x] of polynomials with integer coefficients, nevertheless, the natural number 10001 associated to this polynomial, factors as $73 \cdot 137$, as you can see in Ω_{10001}.

$10001 = x^4 + 1 = 100x^2 + 1 = 99x^2 + 10x + 1 = \ldots = 91x^2 + 88x + 3 \cdot 7 = 7 \cdot 13x^2 + 13 \cdot 3x + 7 \cdot 7x + 3 \cdot 7 = 13x(7x + 3) + 7(7x + 3) = (7x + 3)(13x + 7) = 73 \cdot 137$

Consequently, 110.101.009 factors as $73 \cdot 101 \cdot 109 \cdot 137$, while, $x^8 + x^7 + x^5 + x^3 + x - 1 = (x^2 + 1)(x^2 + x - 1)(x^4 + 1)$.

In the general case: if we call $S(x)$ the standard polynomial of a number a, and $S(x) = R(x)S(x)$ in Ω_a, then $a = R(10)S(10)$. This fact gives us a procedure to get the prime factors of given number or a test to check primality as shown with examples at the previous citation 3, above.

As the example above shows, it is not always true that, $P(x) = R(x)S(x)$ in Ω_a, implies that $P(x)$ factors the same way in **Z**[x]. See also the following

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[6] Ayad, M. et al. *When Does a Given Polynomial with Integer Coefficients Divide Another*? **The American Mathematical Monthly.** Vol. 123, No. 4 April 2016, Pag. 380.

Example

Find the factors, if any, of $f(x) = x^4 + x^3 + 4x^2 + 4x + 9$. This polynomial is the standard polynomial of $a = 11449$. In Ω_a we can establish the following equalities.

$11449 = x^4 + x^3 + 4x^2 + 4x + 9 = 114x^2 + 4x + 9 = \ldots = 100x^2 + 140x + 49 = (10x)^2 + 2(10)(7)x + 7^2 = (10x + 7)^2 = (x^2 + 7)^2 = 107^2$.

The number $11449 = 107 \cdot 107$, however, $f(x)$ is irreducible in Z[x] since all zeros of $f(x)$ are complex, indeed, $x^2 + 7$ has not real roots. We can realize that 107 is a prime number associated to an irreducible polynomial on N[x], the set of polynomials with coefficients in N.

Example

Let us factor the polynomial $f(x) = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x$ inside Ω_a. First of all, we transform this polynomial into a polynomial where all coefficients be nonnegative integers, using the syntactic rule, $x^n = 10x^{n-1}$.

$x^5 - 10x^4 + 35x^3 - 50x^2 + 24x = 10x^4 - 10x^4 + 35x^3 - 50x^2 + 24x = 35x^3 - 50x^2 + 24x = 35x^3 - 5x^3 + 24x = 30x^3 + 2x^2 + 4x = 30240$.

The last polynomial above, can be easily factored, as:

$30x^3 + 2x^2 + 4x = 2(15x^3 + x^2 + 2x) = 2^2(7x^3 + 5x^2 + 6x) = 2^2(7x^2 + 5x + 6)x = 2^3(3x^2 + 7x + 8)x = 2^4(x^2 + 8x + 9)x = 2^4 3(6x + 3)x = 2^4 3^2(2x + 1)x = 2^4 3^3 \cdot 7x = 2^3 3^2(2 \cdot 3)(2 \cdot 5)(7)$.

If we put: $x = 10 = 2 \cdot 5$, $x - 3^2 = 1$, $x - 2^3 = 2$, $x - 7 = 3$, $x - 2 \cdot 3 = 4$. We get: $x = 2 \cdot 5$, $x - 1 = 3^2$, $x - 2 = 2^3$, $x - 3 = 7$, $x - 4 = 2 \cdot 3$.

Replacing above we get: $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x = x(x - 1)(x - 2)(x - 3)(x - 4)$.

The above process, show us how to factor some higher degree polynomials with the recourse of their numerical representation, whenever the polynomial can be factored in **Z**[x]. In this particular case, we have also solved the equation: $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x = 0$. The roots are: $x_1 = 0$, $x_2 =1$, $x_3 = 2$, $x_4 = 3$, and $x_5 = 4$.

Example. We can also solve elementary quadratic and linear equations without any knowledge of algebra. For instance, let us find numbers x such that $f(x) = x^2 - 7x + 10 = 0$.

The first step is to find the natural number associated to f. In Ω_a we can transform f as a polynomial with nonnegative coefficients and find their factors as in the above example.

$x^2 - 7x + 10 = 10x - 7x + 10 = 3x + 10 = 40 = 2^3 \cdot 5 = (10–2)(10–5) = (x - 2)(x - 5)$.

We can verify easily that 2 and 5 satisfy the equation.

Example

A cubic equation like $6x^3 + 25x^2 + 22x + 3 = 0$ has roots in **Q** and can be solved using the standard polynomial.

The first step in order to solve the equation is to find the standard polynomial associated to the above polynomial in the class Ω_a of equivalent polynomials module *a*.

$6x^3 + 25x^2 + 22x + 3 = 8x^3 + 5x^2 + 22x + 3 = 8x^3 + 5x^2 + 2x^2 + 2x + 3 = 8x^3 + 7x^2 + 2x + 3 = 8723$

The polynomial associated to the number 8723 help us to solve the equation.

The second step in the procedure is factoring the standard polynomial in Ω_a, as in the previous examples.

$8723 = 8x^3 + 7x^2 + 2x + 3 = 87x^2 + 2x + 3 = \ldots = 77x^2 + 102x + 3 = 7\cdot 11x^2 + 99x + 33 = 11(7x^2 + 9x + 3) = 11(79x + 3) = 11(78x + 13) = 11(6\cdot 13x + 13) = 11\cdot 13\,(6x + 1) = 11\cdot 13\cdot 61 = (x + 1)(x + 3)(6x + 1)$.

Therefore

$6x^3 + 25x^2 + 22x + 3 = (x + 1)(x + 3)(6x + 1) = 0$. So, the polynomial vanishes when:
$x = -1$, $x = -3$, $x = -1/6$.

The above examples show some applications in connection with a new definition of natural numbers using polynomials in its most primitive form. I hope next time to show you more interesting applications in number theory and algebra.

Armenia, Colombia, December 2016. (First Draft, with some corrections made in January 2017)